Non Linear Dynamics in Einstein-Friedman Equations

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Abstract

Einstein-Friedman equations for the dynamics of a spatially homogeneous and isotropic universe are rederived by constructing a corresponding metric. Dynamics is studied for universes with zero cosmological constant and solutions are characterized as open and closed. Approximate solutions are obtained for non vanishing value of cosmological constant and their behavior is studied. Fixed points of the equations are found and the system is shown to undergo a pitchfork bifurcation at \( \lambda = \kappa \rho \).

1 Introduction

Most of the cosmological models are based on the Copernican principle:

Universe is pretty much the same everywhere.

This in mathematical language translates to a manifold[1] having isotropy[1] and homogeneity[1]. General theory of relativity unifies space and time into a four dimensional manifold. To model the universe the corresponding metric should have spatially isotropic and homogeneous. The dynamical equations for the behavior of space can be obtained from Einstein Field equations. These equations are non-linear and are not exactly solvable for arbitrary values of parameters involved. However the behavior of the solutions can be predicted by analysing the solutions and making physical arguments. Based on these equations and the present state of the universe, its past and fate can also be explored.

2 Deriving Equation

2.1 The Metric

Isotropy and homogeneity implies that space is maximally symmetric. So the metric can be written as:

\[
ds^2 = -dt^2 + a(t)\gamma_{ij}dx^i dx^j
\]

(1)

Where \( a(t) \) is the scale factor which determines at a given slice in time, how big is the universe. \( \gamma_{ij} \) is the metric for a three dimensional maximally symmetric space. The Ricci tensor for such a metric is:
Where $k$ is the Ricci curvature associated to the space. Since maximum symmetry requires the spherical symmetry, we incorporate this by writing the spatial part of (1) in a spherically symmetric manner:

$$d\sigma^2 = \gamma du^i du^j = e^{2\beta(r)}dr^2 + r^2 d\Omega^2 = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$  \hspace{1cm} (3)

The Ricci tensor can be computed using:

$$R_{ij} = \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl} \Gamma^l_{ji} + \Gamma^k_{jl} \Gamma^l_{ki}$$  \hspace{1cm} (4)

Where $\Gamma^i_{jk}$ is the Christoffel symbol.[1] Using (4) and (2) the parameter $\beta(r)$ is found.

$$\beta(r) = -\frac{\ln(1 - kr^2)}{2}$$  \hspace{1cm} (5)

The metric is invariant under the scaling of the curvature, so only relevant parameter is the sign of curvature.

<table>
<thead>
<tr>
<th>Curvature</th>
<th>Metric</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-dt^2 + dr^2 + r^2 d\Omega^2$</td>
<td>Flat, Minkowski Space</td>
</tr>
<tr>
<td>1</td>
<td>$-dt^2 + d\chi^2 + \sin^2(\chi)d\Omega^2$</td>
<td>Closed, de Sitter space</td>
</tr>
<tr>
<td>-1</td>
<td>$-dt^2 + d\psi^2 + \sinh^2(\psi)d\psi^2$</td>
<td>Open, Anti de Sitter space</td>
</tr>
</tbody>
</table>

2.2 Stress Tensor

Since Universe is not empty and the matter in it governs how it evolves, we have to model the matter present in the universe. It is approximated as a perfect fluid of pressure $p$ and density $\rho$. The corresponding stress tensor for a perfect fluid is:

$$T_{\mu\nu} = (p + \rho)U_\mu U_\nu + pg_{\mu\nu}$$

Where $g_{\mu\nu}$ is the metric of the spacetime and $U^\mu$ is the four velocity in co-moving coordinates.[1]

$$U^\mu = (1 \ 0 \ 0 \ 0)$$

By using the Relativistic analogue of Conservation of Energy, equation of motion of the density of matter present in the universe is derived.

$$\Delta_\mu T^\mu_\nu = 0 \implies \dot{\rho} = \frac{3\dot{a}}{a}(p + \rho)$$  \hspace{1cm} (6)
2.3 Einstein Equation

Now I use Einstein Field equation to write down the differential equation satisfied by the scale factor of the universe. Einstein equation is:

\[ R_{\mu\nu} = 8\pi G T_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} \]  \hspace{2cm} (7)

Where \( R \) is the Ricci scalar and \( \Lambda \) is the cosmological constant associated to the energy density of the vacuum.

Let \( 8\pi G = A \), then the temporal part of (7) for \( \mu = \nu = 0 \) yields:

\[ 3 \frac{\dot{a}}{a} + 3 \frac{k}{a^2} - \Lambda - A \rho = 0 \]  \hspace{2cm} (8)

And all the spatial parts give:

\[ \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} + \frac{A}{2} (p + \rho) = 0 \]  \hspace{2cm} (9)

Eliminating \( k \) from the previous two equation I got:

\[ \frac{\ddot{a}}{a} - \frac{\Lambda}{3} + \frac{A}{6} (3p + \rho) = 0 \]  \hspace{2cm} (10)

and by eliminating \( \Lambda \) I got a first order equation:

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{A}{3} \rho - \frac{k}{a^2} \]  \hspace{2cm} (11)

So finally we have our system of equations governing the large scale dynamics of the spacetime:

\[ \begin{pmatrix} \frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{A}{6} (3p + \rho) \\ \left( \frac{\dot{a}}{a} \right)^2 = \frac{A}{3} \rho - \frac{k}{a^2} \\ \frac{\dot{\rho}}{a} = -\frac{3\dot{a}}{a} (p + \rho) \end{pmatrix} \]  \hspace{2cm} (12)

3 Analysis in Universe with zero cosmological constant

By setting \( \Lambda = 0 \), the system (12) reduces to:

\[ \begin{pmatrix} \frac{\ddot{a}}{a} = -\frac{A}{6} (3p + \rho) \\ \left( \frac{\dot{a}}{a} \right)^2 = \frac{4}{3} \rho - \frac{k}{a^2} \\ \frac{\dot{\rho}}{a} = -\frac{3\dot{a}}{a} (p + \rho) \end{pmatrix} \]  \hspace{2cm} (13)

Let

\[ Hubble \ parameter = \ H = \frac{\dot{a}}{a} \]

\[ Deceleration \ parameter = -\frac{a\ddot{a}}{a^2} \]

\[ Density \ parameter = D = \frac{A \rho}{3H^2} = \frac{\rho}{\rho_c} \]

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In terms of these new parameters:

\[ D - 1 = \frac{k}{H^2 a^2} \]

Hence sign of \( k \) is determined by the density parameter and hence the energy density of the matter present in the universe. Curvature is negative, zero and positive for \( \rho < \rho_c \), \( \rho = \rho_c \) and \( \rho > \rho_c \) respectively.

### 3.1 Present State

Observations indicate that the galaxies are moving away from each other in all directions so \( \dot{a} > 0 \), for positive energy density and non negative pressure (13) implies \( \ddot{a} < 0 \). So the present state of the universe can be described as:

- Universe is expanding.
- The expansion is decelerating as it should be expected because gravity acts against expansion.

### 3.2 History and Big Bang

Since \( \ddot{a} < 0 \), we conclude that expansion was faster in the past. If we extrapolate the information about the present state of the universe in past we meet a singularity at \( a = 0 \) where (13) becomes undefined.

#### 3.2.1 Age of Universe

If we have \( \ddot{a} = 0 \), then universe would have expanded linearly with time and its age would be \( \frac{1}{H} \), where \( H_0 \) is the present value of the Hubble parameter. The fact that \( \ddot{a} < 0 \) indicates that the universe is younger than that.
3.2.2 Big Bang

The singularity at $a = 0$, is called the “Big Bang”. It means that the universe came out of a singular state of the spacetime and not by the explosion of matter into pre-existing spacetime. Indeed all cosmological models with positive energy density and non-negative pressure requires that universe must have started from a singularity. As $a \rightarrow 0$, the energy density $\rho \rightarrow \infty$. So classical theory of General Relativity breaks down near the big bang and to understand the state of the universe very close to its beginning a consistent theory of Quantum gravity is required which can explain the structure of spacetime at very tiny distance scales.

3.3 Future and the Fate

The Expansion in the future depends on the curvature.

3.3.1 Open Universes (k=0,-1)

In this case

$$\dot{a}^2 = \frac{A}{3} \rho a^2 + |k| > 0$$

This along with the fact that presently $\dot{a} > 0$ implies that $\dot{a}(t) > 0$ for all future time. So the universes with non-positive curvature will keep on expanding forever and hence the name “open universes”.

Using (13) it is established that $\frac{d}{dt}(\rho a^2) < 0$. Since $\rho a^2 \geq 0$ so as universe keeps on expanding forever and $a \rightarrow \infty$:

$$\rho a^2 \rightarrow 0 \implies \dot{a}^2 \rightarrow |k|$$

So as time marches on the expansion of the universe slows down. In case of zero curvature the universe keeps on expanding at a slower and slower rate and in case of $k = -1$, the expansion rate approaches a non zero value.

3.3.2 Closed Universes (k=1)

The relevant equation in this case is:

$$\dot{a}^2 = \frac{A}{3} \rho a^2 - 1$$

As $a \rightarrow \infty$,

$$\rho a^2 \rightarrow 0 \implies \dot{a}^2 \rightarrow -1$$

which is obviously not true. So a universe with positive curvature can’t keep on expanding forever to infinite size, instead it attains a maximum finite size $a_{\text{max}}$.

This maximum size is found by setting $\dot{a} = 0$.

$$a_{\text{max}} = \sqrt{\frac{3}{A\rho}}$$

As $a \rightarrow a_{\text{max}}$, $\ddot{a} \rightarrow -\frac{A}{6} (3\rho + \rho)a_{\text{max}} < 0$
So at \( a = a_{\text{max}}\), \( \dot{a} = 0 \) and \( \ddot{a} < 0 \). Hence after reaching its maximum size, universe will start contracting and will contract to zero size in the future, reaching another singularity. This singularity in the future is called “Big Crunch”. So a universe with positive curvature started, expanded to a maximum finite size and then contracted to zero size, it existed for a finite time with a finite size and hence justified the name “closed universe”.

4 Analysis with Non zero cosmological constant

By making the following substitution

\[ a \rightarrow \frac{1}{x} \]

and then neglecting the terms of the order \( \left( \frac{1}{x} \right)^2 \) and higher we get the new system of equation in variable \( x \) and \( \rho \).

\[
\begin{align*}
\ddot{x} &= -\frac{k}{3} x^3 - (\Lambda + A \rho) x \\
(\dot{x})^2 &= \left( \frac{4p + \Lambda}{3} \right) x^2 - k x^4 \\
\dot{\rho} &= \frac{4}{x} (p + \rho)
\end{align*}
\]

4.1 Fixed Points and Bifurcations

The system has fixed points.

\[ x = 0, x = \pm \sqrt{\frac{\Lambda + A \rho}{3k}} \]

The negative point is not allowed because of the requirement that size of the universe has to be positive.
4.1.1 Zero curvature

For zero curvature universes, if $\Lambda \neq -A\rho$, only one fixed point exists. If $\Lambda = -A\rho$, then these universes are stationary, i.e., their size does not change with time.

4.1.2 +1 curvature

The second fixed point exists if $\Lambda > -A\rho$. In this case the second fixed point corresponds to the maximum size a positive curvature universe can attain. $x = 0$ corresponds to a saddle point in $x - \rho$ plane and the other point is a sink. Clearly a pitchfork bifurcation occur at $\Lambda = -A\rho$.

4.1.3 -1 Curvature

The second fixed point exists for $\Lambda < -A\rho$. It also corresponds to a maximum attainable size in negative curvature. Its a sink. The first point is saddle. The important role of cosmological constant is evident here, two very different geometric space-times would behave exactly the same way for two different values of cosmological constants. A pitchfork bifurcation occur at $\Lambda = -A\rho$. But it differs from the bifurcation in the previous case.

4.2 Approximate Solutions

If we assume pressure to be small then in $x - \rho$ plane the solution is:

$$\rho = Cx^3$$

Where $C$ is the constant and can be found by initial condition.

In $\dot{x} - x$ plane an invariant of the dynamics is constructed:

$$I = \frac{\dot{x}^2}{2} - \frac{k}{8}x^4 + \frac{\Lambda - A\rho}{3}x^2$$

Which indicates the existence of a closed orbit in $\dot{x} - x$. This also indicates that a homoclinic bifurcation can also occur if the orbit collides with the saddle point ($\dot{x} = 0, x = 0$).[2] If $I$ is thought of Energy then the potential introduced is the same as the Duffing potential and a pitchfork bifurcation would occur for negative curvature and $\Lambda < A\rho$.[3]

Summary and Conclusions

Based on Copernican principle and General relativity, field equations are derived. Then for zero cosmological constant and positive energy density the qualitative behaviour is studied. The curvature turned out to be important in deciding the age and fate of the universe. For non-zero cosmological constant the analysis showed that a bifurcation occur for a particular value of the cosmological constant. Finally approximate solutions are constructed in $x - \rho$ plane and a conserved quantity is found in $\dot{x} - x$ plane and hence a closed trajectory.
References

